Maths Notes

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Contents

Not included: Algebra · Trigonometry · Permutations · Combinations and Probability · Statistics and Difference Equations.

1 Matrices

1.1 Know How To

- Add matrices.
- Multiply matrices by scalars.
- Multiply matrices.
- Find determinant of a matrix.
- Invert a 2x2 matrix.
- Solve two simultaneous equations with matrices.
- Solve matrix equations by pre- or post-multiplying by the inverse.
- Calculate power of diagonal matrices. (Using power property $(M^{-1}AM)^n = M^{-1}A^nM$)

1.2 Be Able To Prove

- Power property $(M^{-1}AM)^n = M^{-1}A^nM$
- Addition and subtraction are commutative.
- The product of two given matrices is a diagonal matrix.

1.3 Formulae

Matrix Addition	$\begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} a+e & c+g \\ b+f & d+h \end{pmatrix}$
Matrix Multiplication	$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{pmatrix}$
Determinant of Matrix	If $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then $det(M) = ad - bc$
Inverse of Matrix	If $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then $\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$
Zero Matrix	$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Identity Matrix	$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Power Property	$(M^{-1}AM)^n = M^{-1}A^nM$
Powers of Diagonal Matrices	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

2 Complex Numbers

2.1 Know How To

- Use the following powers of i:
  - $i = \sqrt{-1}$
  - $i^2 = -1$
  - $i^3 = -i$
  - $i^4 = 1$
- Get the real $Re(z)$ and imaginary $Im(z)$ parts of a complex number.
- Add and subtract complex numbers.
- Multiply complex numbers.
- Get the conjugate of a complex number.
- Divide complex numbers using the conjugate.
- Solve equations by equating real with real and imaginary with imaginary coefficients.

- Plot Argand diagrams using real coefficient as x and imaginary coefficient as y.
- Give the modulus of a complex number. $|z| = |a + bi| = \sqrt{a^2 + b^2}$
- Solve quadratic equations with complex roots using $-b\pm$ formula.
- Express complex numbers under a square root in the form $a + bi$.
- Divide a quadratic into a complex cubic.
- Solve cubic equations with complex roots.
- Write complex numbers in polar form using modulus and argument (angle).
- Use De Moivre's Theorem to:
  - Find powers of complex numbers. (Write in polar form, apply De Moivre's)
  - Prove trigonometric identities. (Use De Moivre's and Binomial Theorem)
  - Find roots of complex numbers. (Write in polar form, general polar form and apply De Moivre's)
- Use De Moivre's Theorem to show $z^n + \frac{1}{z^n} = 2\,cos\,n\theta$ where $z = cos\,\theta + i\,sin\,\theta$ (2005:P1:Q3:C)

2.2 Be Able To Prove

- De Moivre's Theorem $[r(cos\,\theta + i\,sin\,\theta)]^n = r^n(cos\,n\theta + i\,sin\,n\theta)$
- The sum or product of z with its conjugate $\bar{z}$ is always a real number.

2.3 Formulae

De Moivre's Theorem	$[r(cos\,\theta + i\,sin\,\theta)]^n = r^n(cos\,n\theta + i\,sin\,n\theta)$
Modulus Of Complex Number	$\|z\| = \|a + bi\| = \sqrt{a^2 + b^2}$
Roots Of Quadratic	$z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Complex Polar Form	$z = r(cos\,\theta + i\,sin\,\theta)$
General Polar Form	$z = r\,[cos\,(\theta + 2n\pi) + i\,sin\,(\theta + 2n\pi)]$

3 Sequences & Series

3.1 Know How To

- Write the general term of an arithmetic series.
- Write the sum of an arithmetic series.
- Write the sum of a geometric series.
- Evaluate sigma equations.
- Verify equations in u_n by substitution.
- Show a sequence is always increasing or decreasing. ($u_{n+1} > u_n$ or $u_{n+1} < u_n$ for all $n \in \mathbb{N}$)
- Find the value of missing variable(s) in general term given the values of specific terms.
- Be able to convert between sigma and series notation.
- Find the general term u_n when given sum S_n using $u_n = S_n - S_{n-1}$
- Use $a-d$, a, $a+d$ etc. to solve equations of arithmetic series.
- Find the value of the sum to infinity of a series.
- Write recurring decimals as fractions using infinite series.
- Find S_n of an arithmetic-geometric series.
- Find the value of the limit as n tends to infinity ($\lim_{n\to\infty}$) of a sequence.
- Resolve partial fractions and use telescoping to find the sum of series of the form $\frac{1}{r(r+2)}$ and $\frac{1}{(r+1)(r+3)}$
- Use the series of powers of natural numbers to evaluate the sum of sequences.
- Split up series into arithmetic and geometric parts to be evaluated separately.
- Find the number of terms in a sequence that are less than a given number. (2007:P1:Q4:C)
- Expand factorials $n!$ and binomial coefficients $\binom{n}{r}$
- Use induction to prove the equation of the general term of a given sequence. (2007:P1:Q4:C)
- Expand binomials and finding missing variables in equations.

- Write the general term of a given binomial for positive integer powers of n.
- Write a fraction such as $\frac{n^3+8}{n+2}$ in the form $an^2 + n + c$ and hence evaluate the sum to a given n. (Using either long division or by factorising sum/difference of two cubes and cancelling) (2001:P1:Q4:C)

3.2 Be Able To Prove

- A given sequence is/is not arithmetic.
- A given sequence is/is not geometric.
- Algebraic addition, i.e. the reverse of resolving into partial fractions;

$$\frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$$

3.3 Formulae

General Arithmetic Term $\quad u_n = a + (n-1)d$

Sum Of Arithmetic Sequence $\quad S_n = \frac{n}{2}[2a + (n-1)d]$

General Geometric Term $\quad u_n = ar^{n-1}$

Sum Of Geometric Sequence $\quad S_n = \frac{a(1-r^n)}{1-r}$

Sigma Notation $\quad S_n = u_1 + u_2 + \cdots + u_n = \Sigma_{r=1}^{n} u_r$

Sum To Infinity $\quad S_\infty = \frac{a}{1-r}$

$$\Sigma_{r=1}^{n} k = k + k + k + \cdots + k = nk$$

$$\Sigma_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$$

$$\Sigma_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1)$$

4 Differentiation

4.1 Know How To

- Differentiate from first principles x^2, x^3, $\sqrt{x}$, $\frac{1}{x}$, $sin\, x$, $cos\, x$

- Differentiate by rule using;

$$\begin{aligned} \frac{d(x^n)}{dx} &= nx^{n-1} \\ \frac{d(ax^n)}{dx} &= nax^{n-1} \end{aligned}$$

- Evaluate derivatives by substituting a given value.
- Find the second derivative by rule.
- Use the product, quotient and chain rules to differentiate products, fractions and powers respectively.
- Use page 41 to differentiate trigonometric functions. If u is function of x then use page 41 and multiply by $\frac{du}{dx}$
- Implicitly differentiate and get $\frac{dy}{dx}$ on its own.
- Use parametric differentiation to find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and hence find $\frac{dy}{dx}$ by division.
- Use page 41 to differentiate inverse trigonometric functions. If u is function of x then use page 41 and multiply by $\frac{du}{dx}$
- Differentiate exponential functions, substituting u for the power of e if necessary.
- Differentiate logarithms using;

$$\frac{d(\,ln\,u\,)}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

- Differentiate equations such as $y = 2^x$ by using natural log on both sides, implicitly differentiating and get $\frac{dy}{dx}$ on its own.
- Find the equation of a tangent to a curve at a given point.
- Find unknown variables in a curve given slope of tangent at a point.
- Find the local maximum, local minimum and point of inflection. (May be asked for saddle point or turning point)
- Find the vertical and horizontal asymptotes of a curve.
- Apply differentiation to rates of change of area, volume, displacement, velocity and acceleration.

- Solve cubic equations using Newton-Raphson Method;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Show that a given equation has a root in a given range e.g. between 0 and 1.
- Prove given transformations, e.g; Show that a given curve is its own image under central symmetry in the point of intersection of its two asymptotes. (2005:P1:Q6:C)
- Show that a given function for real x cannot have a real y between a and b. (1997:P1:Q7:C)

4.2 Be Able To Prove

- The differential rule.
- The quotient rule.
- The product rule.
- The addition rule.
- A given function only has one real root. (Always increasing or always decreasing)
- A given function does not cross the x-axis, i.e. has no real roots.

4.3 Formulae

Differentiation By First Principles		$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
Differential Rule	$y = ax^n$	$\frac{dy}{dx} = nax^{n-1}$
Product Rule	$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient Rule	$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Chain Rule	$y = u^n$	$\frac{dy}{dx} = nu^{n-1} \cdot \frac{du}{dx}$
Differentiation Of Logs	$y = ln\, u$	$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$
Newton-Raphson		$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

5 Induction

5.1 Know How To

- Prove by induction propositions involving;
  - Divisibility.
  - Series.
  - Inequalities.
- Prove propositions involving series with factorials.
- Prove by induction given expressions;
  - Prove $P(n)$ for the smallest value of n given, e.g. $P(0)$ or $P(1)$.
  - Assume $P(k)$ is true.
  - Show that it is true for $P(k+1)$
  - Conclude with: "$P(k+1)$ is true if $P(k)$ is true. Hence, by the principle of mathematical induction, $P(n)$ is true."

5.2 Be Able To Prove

- The Differential Rule by induction.
- The Factor Theorem by induction.
- De Moivre's Theorem by induction.
- The series of Powers of Natural Numbers by induction;

$$\begin{aligned}
\Sigma_{r=1}^{n} k &= k + k + k + \cdots + k = nk \\
\Sigma_{r=1}^{n} r &= 1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1) \\
\Sigma_{r=1}^{n} r^2 &= 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1)
\end{aligned}$$

6 Binomial Theorem

6.1 Know How To

- Expand factorials and go down terms. ($7! = 7 \cdot 6 \cdot 5!$ and $n! = (n) \cdot (n-1) \cdot (n-2)!$)
- Solve equations involving factorials of n.
- Use the ncr formulae to solve equations in n and to evaluate given combinatorials.
- Expand binomials to a given power n where $n \in \mathbb{N}$ noting that;
  - The expansion contains $(n+1)$ terms.
  - If the binomial is a difference $(a-b)$ then the signs will alternate $+, -, +, - \cdots$.
- Show that a term is independent of x, y, p etc. and be able to find that term.
- Express a given function $f(x)$ as ascending powers of x, y, p etc. (Binomial Expansion)
- Write a binomial $(a + \sqrt{b})^n$ in the form $c + d\sqrt{b}$. (Binomial Expansion and simplify)
- Find the values of variables in a binomial expansion by using the ncr formulae.
- Write the general term of a binomial expansion.

- Use the general term to find a specific term in the expansion.
- Find the middle term of a binomial expansion.
- Find the sum of the coefficients in an expansion by letting x, y, p etc. to equal 1, e.g; $(1+x)^{10} \to (1+1)^{10} = 1024$
- Prove that the sum of given binomial coefficients equal a given number/expression, e.g;

$$\text{Prove that: } \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

6.2 Be Able To Prove

- N/A

6.3 Formulae

n-choose-r	(Definition)	$\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$
n-choose-r	(Easier)	$\binom{n}{r} = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$
n-choose-r	(Simplify)	$\binom{n}{r} = \binom{n}{n-r}$
n-choose-0		$\binom{n}{0} = 1$
n-choose-n		$\binom{n}{n} = 1$
Binomial Expansion		$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{n} a^0 b^n$
Binomial General Term		$u_{r+1} = \binom{n}{r} a^{n-r} b^r$

7 Integration

7.1 Know How To

- Use the basic rule of integration;

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

- Multiply out, cancel or factorise to get the integral in the form ax^n since there is no product/quotient/chain rule.
- Use the definite integral;

$$\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a)$$

- Use substitution for high powers or square roots in product, e.g. $\int 2x(x^2+1)^4\,dx$ Let $u = x^2 + 1$
- Integrate the trigonometric functions *sin* and *cos*;

$$\begin{aligned} \int cos(nx+k)\,dx &= \frac{sin(nx+k)}{n} + c \\ \int sin(nx+k)\,dx &= -\frac{cos(nx+k)}{n} + c \end{aligned}$$

- Convert product of *sin* and *cos* to sum or difference. Write larger angle first and use page 9.
- Convert even powers of cos and sin using double angle formulae, then integrate;

$$\begin{aligned} cos^2 A &= \frac{1}{2}(1 + cos\,2A) \\ sin^2 A &= \frac{1}{2}(1 - cos\,2A) \end{aligned}$$

- Integrate exponential functions (e^{ax}) and substitute for power if necessary.
- Integrate fractions which lead to logarithms, i.e. where the top is the derivative of the bottom.
- Integrate equations in the following form, if the coefficient of x^2 is 1 or use substitution;

$$\begin{aligned} \int \frac{dx}{\sqrt{a^2 - x^2}} &= sin^{-1}\frac{x}{a} + c \\ \int \frac{dx}{a^2 + x^2} &= \frac{1}{a}\,tan^{-1}\frac{x}{a} + c \end{aligned}$$

- Complete the square by adding and/or subtracting a number, e.g.;

$$\int \frac{dx}{x^2 - 4x + 29} = \int \frac{dx}{x^2 - 4x + 4 + 25} = \int \frac{dx}{(x-2)^2 + 5^2}$$

and let $u = (x - 2)$ and integrate.

- Substitute $x = a\,sin\,\theta$ in integrals of the form $\int \sqrt{a^2 + x^2}$, eliminate square root, convert limits to θ and integrate.
- Calculate the area under a curve by integrating y within certain limits.
- Calculate volumes of revolution using $\pi \int_a^b y^2\,dx$ (or x^2 for rotation around the y-axis)
- Use integration to derive the formula for volume of a cone/sphere. (Volume of Rotation)

7.2 Be Able To Prove

- N/A

7.3 Formulae

Integration Rule	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$
Integration With Limits	$\int_a^b f^{'}(x)\,dx = [f(x)]_a^b = f(b) - f(a)$

8 Periodic Functions and Limits Of Functions

8.1 Know How To

- State the period and range of a given function or diagram.
- State the period and range of trigonometric functions on a diagram.
- Find $\lim_{x \to n} f(x)$ where $x \in \mathbb{R}$
- Find $\lim_{x \to 0} f(x)$ (Factorise/cancel if bottom will become 0, then substitute 0 for x)
- Find $\lim_{x \to \infty} f(x)$ (Divide all terms by highest power of x)
- Use the trigonometric limit identities to simplify trigonometric equations.

8.2 Be Able To Prove

- N/A

8.3 Formulae

Periodic Function	(h=period)	$f(x+h) = f(x)$
Sine		$\lim_{\theta \to 0} \frac{sin\, n\theta}{n\theta} = \lim_{\theta \to 0} \frac{n\theta}{sin\, n\theta} = 1$
Cosine		$\lim_{\theta \to 0} cos\, \theta = 1$
Tan		$\lim_{\theta \to 0} \frac{tan\, n\theta}{n\theta} = \lim_{\theta \to 0} \frac{n\theta}{tan\, n\theta} = 1$

9 Geometry Of The Circle

9.1 Know How To

- Find radius.
- Find centre.
- Find equation of circle given Point/Tangent/Radius/Centre/Points-as-Diameter.
- Find if a point is inside, outside or on a circle. (Using distance or algebra)
- Find value(s) of k if point $(k, 0)$ lies on given circle C.
- Find value of variable in circle, given radius/diameter/points etc.
- Prove that a locus (equation) is a circle. (May be given other information such as chords/points that are perpendicular/equidistant etc.)
- Prove parametric equations represent locus of a circle;
  - Find Cartesian equation given X and Y in T. (Get T by itself and let equations equal each other.)
  - Find Cartesian equation given X and Y in θ. (Use $cos^2\, \theta + sin^2\, \theta = 1$)
- Find point(s) of intersection between line and circle.
- Find equation of a circle given;
  - 3 points on the circle. (Substitute and solve simultaneous equations of g, f and c)
  - 2 points and a line containing the centre. (Substitute and solve simultaneous equations in g, f and c)
  - 2 points and a tangent, since the perpendicular of a tangent contains the centre.

  - 1 point, radius and a line that contains centre or is tangent.

- Find equation of tangent at a given point. (Find slope from centre to point of tangency)
- Prove that a line is a tangent to a circle. (Use perpendicular distance from center to line)
- Find length of a tangent from a point outside circle. (Distance to point of tangency)
- Find value of variable(s) in point/circle given length of tangent.
- Find equation of tangent(s) given circle and distance/angle/slope to other line(s).
- Find equations of two tangents containing point outside circle. (Perpendicular Distance $\pm$)
- Find equations of circles or prove equations based on axes as tangents. ($g^2 = c$ or $f^2 = c$)
- Show that circles touch externally/internally and find point of contact. (Using division of line segment)
- Find equation of circle given radius/centre knowing it touches another circle.
- Find where a circle intersects the X or Y axis. (Let $y = 0$ or $x = 0$)
- If two circles C_1 and C_2 share a chord or common tangent then $C_1 - C_2 = 0$ is the equation of the chord/tangent.
- Find the equation of a circle given a chord and two of: length of chord/radius/perpendicular distance from chord to centre.
- Show that $|\angle abc|$ is a right angle given three points a, b and c. (2002:P2:Q1:B)
- Find coordinates of the point on a given circle closest to a given line. (2006:P2:Q1:C)

9.2 Be Able To Prove

- A given locus (equation) is a circle.
- A set of parametric equations represent a circle.
- A given line is a tangent to a given circle.
- When circle touches X axis, $g^2 = c$.
- When circle touches Y axis, $f^2 = c$.

- When circle touches X and Y axes, $g^2 = f^2$.
- Two given circles touch externally/internally.
- A given line does not intersect a given circle. (2006:P2:Q1:C)
- Prove the tangent to the circle $x^2 + y^2 = r^2$ at (x_1, y_1) is $xx_1 + yy_1 = r^2$

9.3 Formulae

Circle, centre (0,0)		$x^2 + y^2 = r^2$
Circle, centre (h,k)		$(x-h)^2 + (y-k)^2 = r^2$
Circle, general equation		$x^2 + y^2 + 2gx + 2fy + c = 0$
Trigonometric Parametric		$x = h \pm r\,cos\,\theta$ and $y = k \pm r\,sin\,\theta$
Perpendicular Distance		$d = \dfrac{\lvert ax + by + c \rvert}{\sqrt{a^2 + b^2}}$
Division of Line Segment	(Touch Externally)	$(x, y) = \left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n} \right)$
Division of Line Segment	(Touch Internally)	$(x, y) = \left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n} \right)$

10 The Line And Linear Transformations

10.1 Know How To

- Recall and use all formulae and identities from Junior Cert.
- Transform points under translation.
- Transform points and lines using the following;
  - Axial symmetry in x-axis $\longrightarrow$ change the sign of y.
  - Axial symmetry in y-axis $\longrightarrow$ change the sign of x.
  - Central symmetry in the origin $\longrightarrow$ change the sign of x and y.
- Show two lines are parallel or perpendicular.

- Show that three points are collinear. (Show area is 0)
- Find the area of a triangle given three points. (Translate one point to the origin, translate the others accordingly and use area formula)
- Divide a line segment in a given ratio externally or internally. (Division of line segment formula)
- Find the centroid g, circumcentre o and orthocentre h of a triangle.
- Find the perpendicular distance from a point to a line.
- Find the perpendicular distance between two parallel lines.
- Find the point of intersection of two lines. (Simultaneous equations)
- Show that two points are on the same/opposite side as each other. (Substitution)
- Find the angle between two lines. (Acute and obtuse)
- Find the value(s) of missing variable(s) in two lines given the angle between them. (Angle formula)
- Find the equations of the lines through a given point which make a given angle with another line. (Angle formula $\pm$)
- Find the equations of concurrent lines and find the relationship between λ and μ in a given equation.
- Find the Cartesian equation of a line given parametric equations of x and y.
- Show an equation or two parametric equations represent a line. (Equation is of the form $ax + by + c = 0$)
- Show that two parametric equations represent a given line segment. (Substitute points separately into the parametric equations and the results should be between the given range of t e.g. $-2 \leq t \leq 1$)
- Transform points using $f(x,\, y) = (x^{'}, y^{'})$
- Recall and use the following properties;
  - The origin (0,0) is mapped onto the origin.
  - A line is mapped onto a line.
  - A line segment is mapped onto a line segment.
  - Pairs of parallel lines are mapped onto pairs of parallel lines.
  - Parallelograms are mapped onto parallelograms.

  - In general, pairs of perpendicular lines are **not** mapped onto pairs perpendicular lines.
  - In general, distances and area are **not** invariant (they may change) under linear transformation.

- Transform lines. (Eliminate x in one equation and y in the other and substitute into the line)
- Show lines, triangles, squares etc. on a diagram.

10.2 Be Able To Prove

- Perpendicular distance from a point to a line formula.
- Angle between two lines formula.
- A given transformation maps every pair of parallel lines to a pair of parallel lines. (2005:P2:Q3:C) (2002:P2:Q3:B)

10.3 Formulae

Distance Between Two Points		$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
Midpoint Of A Line Segment		$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
Slope Of A Line		$m=\frac{y_2-y_1}{x_2-x_1}$
Equation Of A Line		$(y-y_1)=m(x-x_1)$
Slope Of A Given Line	$ax+by+c=0$	$m=-\frac{a}{b}$
Area Of a Triangle		$\frac{1}{2}\lvert x_1y_2-x_2y_1\rvert$
Division of Line Segment	(Internal)	$(x,y)=\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$
Division of Line Segment	(External)	$(x,y)=\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$
Distance From A Point To A Line		$d=\frac{\lvert ax+by+c\rvert}{\sqrt{a^2+b^2}}$
Angle Between Two Lines	(Acute Angle)	$tan\,\theta=\left\lvert\frac{m_1-m_2}{1+m_1m_2}\right\rvert$

11 Vectors

11.1 Know How To

- Multiply a vector by a scalar.
- Use linkage to add $\vec{ab} + \vec{bc} = \vec{ac}$
- Represent two-letter vectors as single letters $\vec{ab} = \vec{b} - \vec{a}$
- Find the midpoint of a vector. $\vec{m} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$ where $\vec{m}$ is the vector from the origin to the midpoint m.
- Use the centroid/circumcentre of a triangle to prove vector equations. (2006:P2:Q2:C)
- Use the ratio theorem.
- Equate coefficients of parallel and non-parallel vectors.
- Use dot product. $\vec{x} \cdot \vec{y} = |\vec{x}|\,|\vec{y}| cos\theta$
- Find the angle between two vectors.
- Solve equations of $\vec{i}$ and $\vec{j}$ and plot diagrams of $\vec{i}$ and $\vec{j}$.
- Get the modulus of a vector. $|a\vec{i} + b\vec{j}| = \sqrt{a^2 + b^2}$
- Find the unit vector of $\vec{v}$. $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$
- Get the related perpendicular vector. (Swap coefficients and change sign of $\vec{i}$)
- Write a vector in terms of parallel and perpendicular vectors $\vec{r}$ and $\vec{r}^{\perp}$ to solve for h and k.
- Show that a point is on the bisector of an angle. (2007:P2:Q2:C)

11.2 Be Able To Prove

- The length (modulus) of any unit vector $\frac{\vec{v}}{|\vec{v}|}$ is 1.
- The following vector properties:
  - If $\vec{x} \perp \vec{y}$ then $\vec{x} \cdot \vec{y} = 0$ and vice-verse.
  - $\vec{x} \cdot \vec{x} = |\vec{x}|^2$
  - If $\vec{x} \parallel \vec{y}$ then $\vec{x} \cdot \vec{y} = |\vec{x}|\,|\vec{y}|$
  - $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
  - $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
  - If $\vec{x} = a\vec{i} + b\vec{j}$ and $\vec{y} = c\vec{i} + d\vec{j}$ then $\vec{x} \cdot \vec{y} = ac + bd$

– If θ is the angle between $\vec{x}$ and $\vec{y}$ then:

$$cos\theta = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}|\,|\vec{y}|}$$

11.3 Formulae

Midpoint Of Vector	$\vec{m} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$
Ratio Theorem	$\vec{x} = \frac{n}{m+n}\vec{a} + \frac{m}{m+n}\vec{b}$
Modulus Of Vector	$\lvert a\vec{i} + b\vec{j}\rvert = \sqrt{a^2 + b^2}$
Unit Vector	$\frac{\vec{v}}{\lvert\vec{v}\rvert}$
Dot Product	$\vec{x} \cdot \vec{y} = \lvert\vec{x}\rvert\,\lvert\vec{y}\rvert cos\theta$
Angle Between Two Vectors	$cos\theta = \frac{\vec{x} \cdot \vec{y}}{\lvert\vec{x}\rvert\,\lvert\vec{y}\rvert}$
Related Perpendicular Vector	If $\vec{r} = a\vec{i} + b\vec{j}$ then $\vec{r}^{\perp} = -b\vec{i} + a\vec{j}$

12 Option Question: Further Calculus & Series

12.1 Know How To

- Use differentiation to maximise or minimise the value of a given function.
  - Draw a diagram if necessary.
  - Write the expression in **one** variable by finding a link.
  - Differentiate the expression, let it equal to 0.
  - Check value is a max or min.
  - Substitute if necessary to give final answer.
- Integrate by parts using the formula on page 42 of the maths tables;

$$\int u\,dv = uv - \int v\,du + c$$

- Evaluate integration by parts with limits.
- Use the ratio test to determine if a series is convergent, divergent or the result is inconclusive.

- Use the ratio test to determine if a power series is convergent for a range of values of x, all x or only $x = 0$.
- Derive the Maclaurin Series.
- Write the Maclaurin Series for the following six functions;

  Exponential $f(x) = e^x$

  Logarithmic $f(x) = ln\,(1 + x)$

  Sine $f(x) = sin\,x$

  Cosine $f(x) = cos\,x$

  Binomial $f(x) = (1 + x)^n$

  Inverse Tan $f(x) = tan^{-1}\,x$

- Use the Maclaurin Series to approximate;
  - e
  - $ln\,a$ where $a \in \mathbb{R}$
  - $cos\,a$ where $a \in \mathbb{R}$
  - $sin\,a$ where $a \in \mathbb{R}$
  - $tan^{-1}\,a$ where $a \in \mathbb{R}$
  - π, PI
  - $\sqrt{a}$ or $(1 + a)^n$ where $a \in \mathbb{R}$ and $n \in \mathbb{N}$ using Binomial Expansion.
- Write the general term of the Maclaurin series of a given function.
- Derive the Maclaurin Series for $tan^{-1}\,x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$
- Approximate PI using $tan^{-1}\,x$
- Find the missing fraction a or b in an equation such as $tan^{-1}a + tan^{-1}b = \frac{\pi}{4}$

12.2 Be Able To Prove

- $tan^{-1}\,a + tan^{-1}\,b = tan^{-1}\,\left(\frac{a+b}{1-ab}\right)$

12.3 Formulae

Integration By Parts	$\int u\,dv = uv - \int v\,du + c$
Ratio Test	$R = \lim_{n\to\infty} \left\lvert \frac{u_{n+1}}{u_n} \right\rvert$
Maclaurin Series	$f(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$
Maclaurin Series General Term	$u_n = \frac{f^{(n)}(0)}{n!}x^n$
General Binomial Coefficient	$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$